Name: .... Maths Class: ....

SYDNEY TECHNICAL HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE
2003

Mathematics

TIME ALLOWED: 3 hours plus 5 mins reading time.

Instructions:

- Write your name and class at the top of this page, and at the top of each answer sheet.
- At the end of the examination this examination paper must be attached to the front of your answers.
- All questions are of equal value and may be attempted.
- All necessary working must be shown. Marks will be deducted for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary.

(FOR MARKERS USE ONLY)

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	TOTAL

QUESTION 1:

(a) Give the exact value of $\cos\frac{7\pi}{6}$ **1**

(b) Simplify $(\sqrt{2}+1)^2+(\sqrt{2}-1)^2$ **2**

(c) Simplify $\frac{1}{x^2-1}-\frac{1}{x+1}$ **2**

(d) Solve the following inequality, and plot the solution on a number line. **2**

$$1-3x<7$$

(e) Find the value of $e^{-\frac{\pi}{2}}$ correct to 4 decimal places **1**

(f) Find the values of a and b if $a+b\sqrt{5} = 9-\sqrt{80}$ **2**

(g) If $f(x)=\begin{cases}1-2x & \text{for } x\le 0\\ x & \text{for } x>0\end{cases}$

find (i) $f(3)+f(-1)$ **2**

(ii) $f(a^2)$

QUESTION 2: (Begin on a new page)

(a) Differentiate with respect to x: (i) $x\sqrt{x}$ **1**

(ii) $2\cos x\sin x$ **1**

(iii) $\log\left(\frac{x+1}{x-1}\right)$ **2**

(b) Find indefinite integrals of: (i) $\frac{x^2}{x^3-2}$ **1**

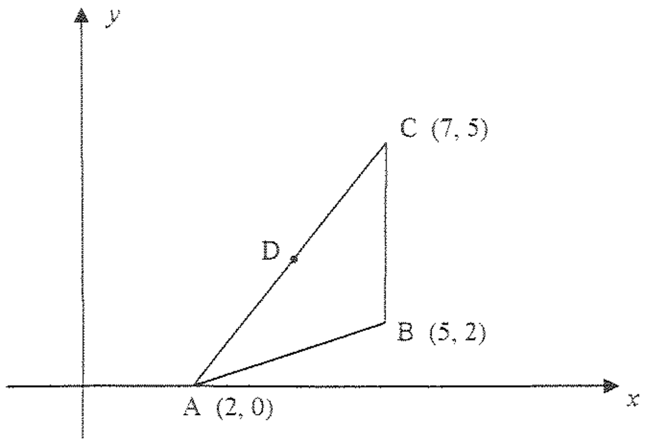
(ii) $\cos\frac{\pi x}{3}$ **2**

(iii) $\frac{5}{(2x-1)^2}$ **2**

(c) Find the exact value of $\int_0^{\frac{\pi}{2}} 3\sec^2\frac{3x}{2}\,dx$ **3**

QUESTION 3: (Begin on a new page)

(a) The points A(2,0), B(5,2) and C(7,5) are joined to form a triangle as shown below. D is the midpoint of AC.



(i) Find the length of AC **1**

(ii) Find the co-ordinates of D **1**

(iii) Find the slope of DB, and prove that DB is perpendicular to AC **2**

(iv) BD is extended to E, so that BD = DE.
Find the co-ordinates of the point E **1**

(v) Find the area of the quadrilateral ABCE **2**

(b) If $y = e^{x^2}$, find (i) $\frac{dy}{dx}$ **1**

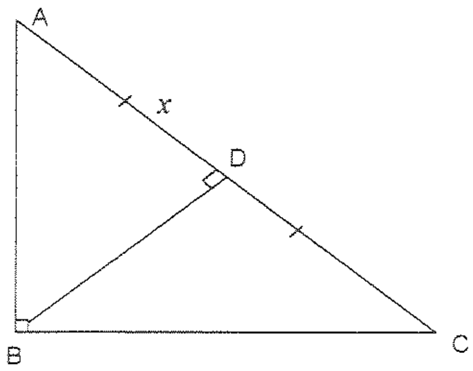
(ii) $\frac{d^2y}{dx^2}$ **1**

(c) Find the co-ordinates of the vertex of the parabola **3**

$$12y - x^2 + 4x - 16 = 0$$

QUESTION 4: (Begin on a new page)

(a) Copy the following diagram neatly onto your answer sheet:



Δ ABC is right-angled at B.

BD $\perp$ AC

D is the midpoint of AC

AD is x units in length.

(i) Prove that Δ ABD is similar to Δ ACB **2**

(ii) Prove that $AD.AC = AB^2$ and hence find the length of AB in terms of x. **2**

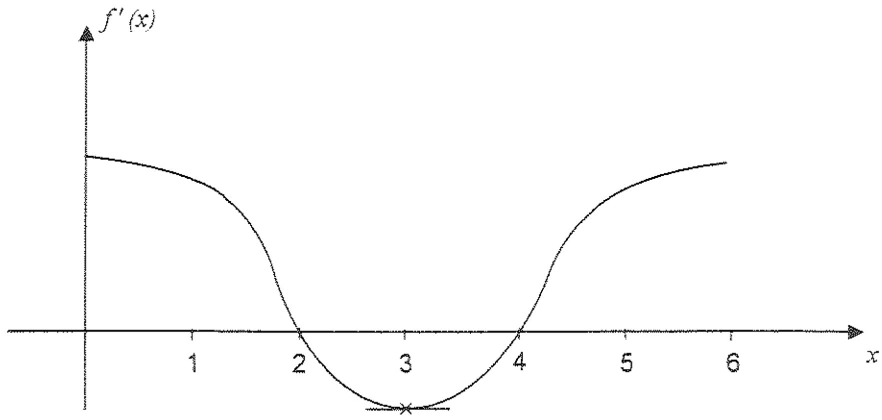
(iii) Show that $\angle BAD = 45°$ **2**

(iv) Hence, find the ratio $\dfrac{AB}{BC}$. **2**

(b) Find the equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3,4) **4**

QUESTION 5 : (Begin on a new page)

(a) The curve of $y = f'(x)$ for a certain function $f(x)$ is shown below for the domain $0 \le x \le 6$.



It is also known that $f''(2) < 0$, $f''(4) > 0$, and that $f(0) = f(6) = 5$

On your answer page, draw a possible graph of $y = f(x)$ for $0 \le x \le 6$ which incorporates all you may deduce about the function $f(x)$. **4**

(b) Given that $y = x^3 - 9x^2 + 24x$,

(i) find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ **1**

(ii) find all stationary points and determine their nature **4**

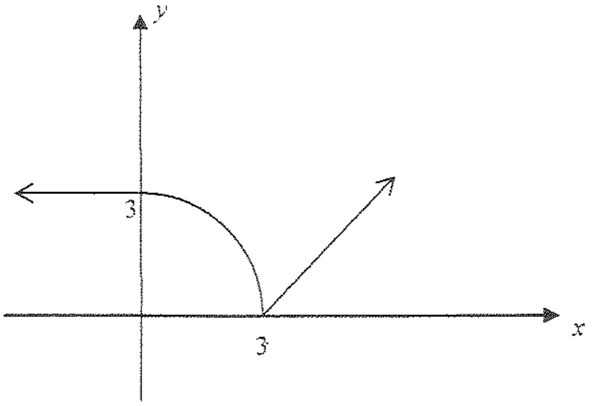
(iii) find the point of inflexion **1**

(iv) sketch the curve, showing all important features **2**

QUESTION 6: (Begin on a new page)

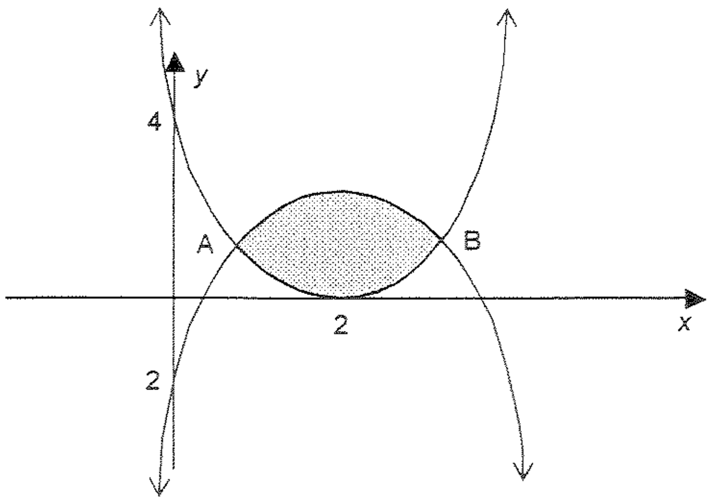
(a) The sketch below is of a function defined by

$$g(x) = \begin{cases} 3 & \text{for } x < 0, \\ \sqrt{9 - x^2} & \text{for } 0 \le x \le 3, \\ x - 3 & \text{for } x > 3 \end{cases}$$



Find the exact value of $\int_{-1}^{4} g(x)\,dx$ 3

(b) The curves $y = (x-2)^2$ and $y = 4x - x^2 - 2$ are shown at below and intersect at the points A and B



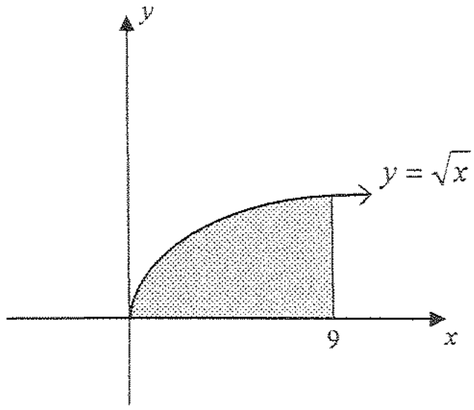
(i) Find the x-values of the points A and B 2

(ii) Find the shaded area. 3

QUESTION 6 CONTINUES OVERPAGE…)

QUESTION 6 (continued)

(c) Find the volume of the solid formed when the shaded area under the curve $y=\sqrt{x}$, shown at right, is rotated around the **y-axis** 4



<u>QUESTION 7</u>: (Begin on a new page)

(a) (i) Explain why the equation $2x^2-x+3=0$ has no real roots 1

(ii) You are given that the roots of $2x^2-x+3=0$ are α and β.

Find the value of (I) $\alpha+\beta$ 1

(II) $\alpha\beta$ 1

(III) $\alpha^2+\beta^2$ 1

(IV) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$ 1

(iii) Form another quadratic equation with roots x_1 and x_2 where $x_1=\dfrac{1}{\alpha}$ 2

and $x_2=\dfrac{1}{\beta}$ where α and β are the roots referred to in part (ii) above.

(b) King Megarich wished to save for his daughter's wedding so he decided to put some money each week onto a chessboard. Each week was allocated to a different square on the board; 64 weeks in all. He put 10 cents on the board each week for the first 4 weeks. For the next 4 weeks he put 20 cents on the board each week. For the following 4 weeks he put down 40 cents per week and he continued following this pattern until he had completely filled the chessboard. How much, to the nearest dollar, had he saved for the wedding? 2

(c) Find the algebraic equation giving the locus of a point which moves so that its distance from the point (0, 2) is always three times its distance from the line $y=-2$. 3

QUESTION 8: (Begin on a new page)

(a) The position in metres of a particle moving along the x-axis after t seconds is given by the equation

$$x = 40 + 10t - 5t^2$$

(i) Find its initial position. 1

(ii) At what time (in seconds) is the particle at the origin? 1

(iii) When does the particle come to rest? 1

(iv) Describe the acceleration of the particle. 1

(v) What is its velocity after (α) $\frac{1}{2}$ second? (β) 2 seconds? 2

(vi) What happens to the particle between the times $t = {}^1/_2$ sec and $t = 2$ secs? 1

(b) Solve the equation $4\sin^2\theta - 3 = 0$ for $0 \le \theta \le 2\pi$. 4

(c) Find the value of $\sum_1^\infty \left(\frac{2}{3}\right)^n$ 1

QUESTION 9: (Begin on a new page)

(a) A cylindrical can with an open top has a base radius of r cm and a height of h cm. It is to be made using 75π cm^2 of tin.

(i) Show that $h = \dfrac{75 - r^2}{2r}$ 2

(ii) Find an expression for the volume of the can, V, in terms of r. 1

(iii) Show that the maximum volume of this can is 125π cm^2. Find the height and base radius of such a tin can. 5

(b) (i) Sketch the curve $y = 3\sin\dfrac{\pi x}{2}$ for $-2 \le x \le 4$. 1

(ii) Draw on your diagram a line, clearly labelled, which can be used to solve the following equation: 2

$$\sin\frac{\pi x}{2} - \frac{x}{3} = 0$$

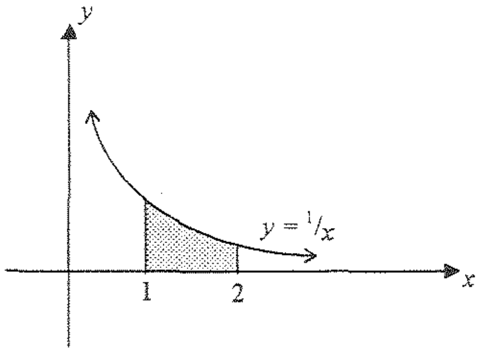
(iii) Determine the number of solutions to the equation 1

$$\sin\frac{\pi x}{2} - \frac{x}{3} = 0 \quad \text{over the domain } -2 \le x \le 4,$$

DO NOT ATTEMPT TO SOLVE THIS EQUATION.

QUESTION 10: (Begin on a new page)

(a)



(i) By using one application of Simpson's Rule (ie 3 function values), find an approximation for the area shaded above, leaving your answer as a fraction in simplest terms. **2**

(ii) Using integration, find an expression for the exact area shaded above. **2**

(iii) Using your answers to parts (i) and (ii) above and the fact that $e^{\ln 2} = 2$, find an approximation for e correct to 4 decimal places. **2**

(b) Prove that the curve $y = x^2 \ln x$ is concave upwards for $x > e^{-\frac{3}{2}}$ **3**

(c) Find $\frac{d}{dx} e^{\cos x}$ and hence show that $\int_0^{\frac{\pi}{2}} e^{\cos x} \sin x \, dx = e - 1$ **3**

End of paper.

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SOLUTIONS – 2003 2 UNIT TRIAL

QUESTION 1:

(a) $-\sqrt{3}/2$ (1)

(b) $(3+2\sqrt{2})+(3-2\sqrt{2})$ (1)

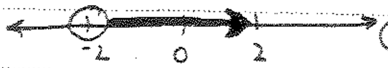
$= 6$ (1)

(c) $\frac{1}{(x+1)(x-1)} - \frac{x-1}{(x+1)(x-1)}$ (1)

$= \frac{2-x}{(x+1)(x-1)}$ (1)

(d) $3x > -6$

$x > -2$ (1)



(1)

(e) 0.2079 (1)

(f) $9-\sqrt{80} = 9-4\sqrt{5}$

$\therefore \begin{cases} a = 9 & (1) \\ b = -4 & (1) \end{cases}$

(g) (i) $f(3)+f(-1) = 3+3$

$= 6$ (1)

(ii) $f(a^2) = a^2$ (1)

QUESTION 2:

(a) (i) $\frac{3}{2}x^{1/2}$ OR $\frac{3\sqrt{x}}{2}$ (1)

(ii) $2\sin x(-\sin x) + 2\cos x\cos x$

$= 2\cos^2 x - 2\sin^2 x$ (1) either

(iii) $\frac{d}{dx}\{\log(x+1) - \log(x-1)\}$ (1)

$= \frac{1}{x+1} - \frac{1}{x-1}$ (1)

(b) (i) $\frac{1}{3}\log_e(x^3-2) + k$ (1)

(ii) $\frac{3}{\pi}\sin\frac{\pi x}{3} + k$

(1) (1)

(iii) $-\frac{5}{2}(2x-1)^{-1} + k$

OR $-\frac{5}{2(2x-1)} + k$

(2) 1 off for each bit wrong, accept equivalent answers

1 off if no constants are shown in all questions

2 (c) $2\tan\frac{3x}{2}\Big]_0^{\pi/2} = 2\tan\frac{3\pi}{4} - 2\tan 0$ (1) (1)

$= -2$

QUESTION 3:

(a)(i) $AC = \sqrt{(7-2)^2 + 5^2}$

$= \sqrt{50}$ OR $5\sqrt{2}$ — either (1)

(ii) D is $\left(\frac{9}{2}, \frac{5}{2}\right)$ (1)

(iii) $m_{DB} = \frac{1/2}{-1/2}$

$= -1$ (1)

$m_{AC} = 5/5 = 1$

$\therefore m_{AC} \cdot m_{DB} = -1$

$\therefore AC \perp DB$ (1)

(iv) $E = (4, 3)$ (1)

(v) ABCE is a kite since AE and BE intersect at 90° / bisect.

$\therefore$ Area $= \frac{1}{2} AC \times BE$ (

AND $BE = \sqrt{(1)^2 + (1)^2}$

$= \sqrt{2}$

$\therefore$ Area $= \frac{1}{2} \times 5\sqrt{2} \times \sqrt{2}$

$= 5\ u^2$ (1)

(b) $y = e^{x^2}$

(i) $\frac{dy}{dx} = 2xe^{x^2}$ (1)

(ii) $\frac{d^2y}{dx^2} = 2e^{x^2} + 2x.2xe^{x^2}$

$= 2e^{x^2}[1+2x^2]$ either (1)

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QUESTION 3 (i)

$12y = x^2 - 4x + 16$

$12y = (x-2)^2 + 12$

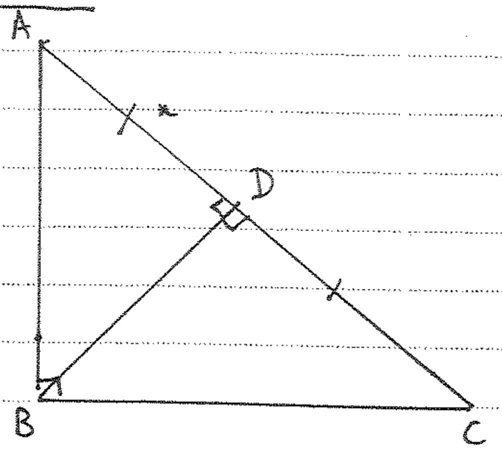
$12(y-1) = (x-2)^2 \leftarrow (1)$

$(x-2)^2 = 4 \cdot 3 \cdot (y-1) \leftarrow (1)$

$V: (2,1) \leftarrow (1)$

QUESTION 4:

(a)



(i) In $\triangle ABD$ and $\triangle ACB$,

$\angle A$ is common (1)

$\angle ADB = \angle ABC = 90°$ (1)

$\therefore \triangle ABD \ ||| \ \triangle ACB$ (equiangular)

(ii) Because of similarity,

$AD/AB = AB/AC$

$\therefore AD \cdot AC = AB^2$ } (1)

$\therefore x \cdot 2x = AB^2$

$\therefore AB = x\sqrt{2}$ (1)

(iii) Using Pythagoras in $\triangle ABD$,

$BD^2 = AB^2 - AD^2$

$= 2x^2 - x^2$

$\therefore BD = x$

$\therefore \triangle ABD$ is isosceles,

$\therefore \angle BAD = 45°$ (angle sum)

(2) OR // (2)

$\cos \angle BAD = AD/AB$

$= x/x\sqrt{2}$

$= 1/\sqrt{2}$

$\therefore \angle BAD = 45°$

(iv) If $\angle BAD = 45°$, then $\triangle ABC$ is isosceles (1)

$\therefore AB/BC = 1$ ... (2)

4 (b) $x^2 + y^2 = 25$

$y = \sqrt{25 - x^2}$ (1)

$\therefore \frac{dy}{dx} = \frac{1}{2}(-2x)(25-x^2)^{-\frac{1}{2}}$ (1)

At (3,4) $m_T = \frac{-3}{4}$ (1)

Equation is:

$y - 4 = -\frac{3}{4}(x-3)$

$4y - 16 = -3x + 9$

$3x + 4y - 25 = 0$ (1)

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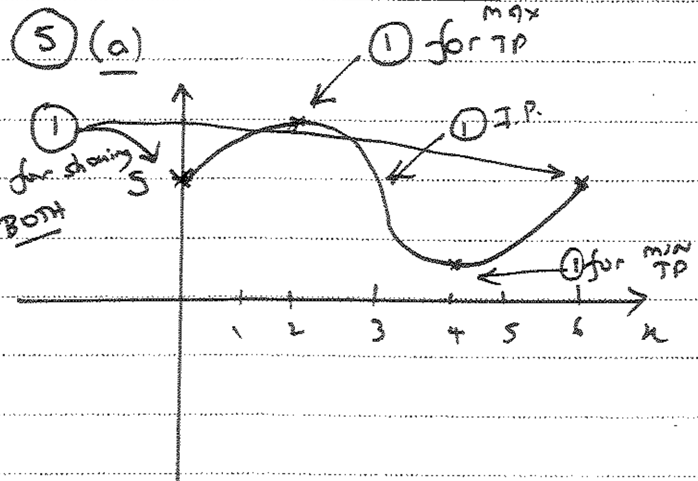
5 (a)



(b) $y = x^3 - 9x^2 + 24x$,

(i) $\frac{dy}{dx} = 3x^2 - 18x + 24$, $\frac{d^2y}{dx^2} = 6x - 18$ (1)

(ii) At S.P.'s $\frac{dy}{dx} = 0$ (1)

$\therefore x^2 - 6x + 8 = 0$

$(x-4)(x-2) = 0$ (1)

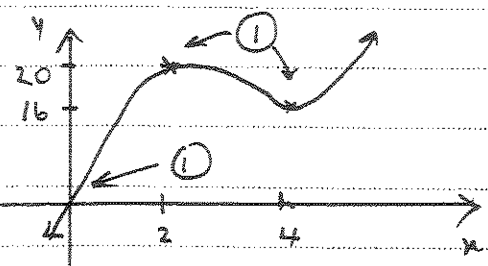
$\therefore \begin{cases} x = 4 \\ y = 16 \\ y'' > 0 \end{cases}$ or $\begin{cases} x = 2 \\ y = 20 \\ y'' < 0 \end{cases}$

$\therefore$ MIN T.P. at $(4, 16)$ (1) MAX T.P. at $(2, 20)$ (1)

(iii) At I.P. $\frac{d^2y}{dx^2} = 0$

$\therefore \begin{cases} x = 3 \\ y = 18 \end{cases}$ (1) accept just $x = 3$

(iv)



6

(a) $A_1 = \int_{-1}^{0} g(x)\,dx$

$= 1 \times 3 = 3u^2$ (1)

$A_2 = \int_0^3 g(x)\,dx$

$= \frac{1}{4} \times \pi \times 9$ (1)

$= \frac{9\pi}{4}$

$A_3 = \int_3^4 g(x)\,dx$

$= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$ (1)

$\therefore$ Area $= 3\frac{1}{2} + \frac{9\pi}{4}$.

(b)(i) $(x-2)^2 = 4x - x^2 - 2$ (1)

$x^2 - 4x + 4 = 4x - x^2 - 2$

$\therefore 2x^2 - 8x + 6 = 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$\therefore x = 3$ or $x = 1$ (1)

(ii) $A_1 = \int_1^3 4x - x^2 - 2\,dx$

$= \left[2x^2 - \frac{1}{3}x^3 - 2x\right]_1^3$

$= (18 - 9 - 6) - (2 - \frac{1}{3} - 2)$

$= 3\frac{1}{3}u^2$ (1)

$A_2 = \int_1^3 x^2 - 4x + 4\,dx$

$= \left.\frac{1}{3}x^3 - 2x^2 + 4x\right]_1^3$

$= (9 - 18 + 12) - (\frac{1}{3} - 2 + 4)$

$= \frac{2}{3}u^2$ (1)

$\therefore$ Area $= 3\frac{1}{3} - \frac{2}{3}$

$= 2\frac{2}{3}u^2$ (1)

OR

$\int_1^3 (2x^2 - 8x + 6)\,dx$ (1)

$= \left[\frac{2}{3}x^3 - 4x^2 + 6x\right]_1^3$ (1)

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6 (c) $Vol_{cylinder} = \pi r^2 h$

$= \pi (9)^2 (3)$ (1)

$= 243\pi \quad (763.41)$

Vol under curve $= \pi \int_0^3 x^2 \, dy$ (1)

$= \pi \int_0^3 y^4 \, dy$

$= \pi \left[\frac{1}{5} y^5\right]_0^3$

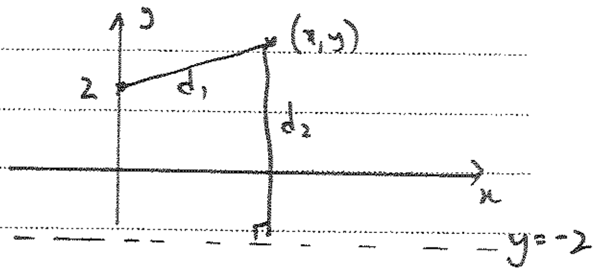
$= \pi \cdot \frac{243}{5}$ (1)

$= \frac{243\pi}{5} \quad (152.68)$

$\therefore$ Vol $= 243\pi - \frac{243\pi}{5}$

$= \begin{cases} \frac{972\pi}{5} \\ 610.73 \ u^3 \end{cases}$ (1)

(c)



$\left.\begin{array}{l} d_1 = \sqrt{x^2 + (y-2)^2} \\ d_2 = y + 2 \end{array}\right\}$ (1)

$d_1 = 3 d_2$ (1)

$d_1^2 = 9 d_2^2$

$x^2 + y^2 - 4y + 4 = 9y^2 + 36y + 36$

$\therefore 8y^2 + 40y - x^2 + 32 = 0$

(or equivalent) (1)

7 (a)(i) $\Delta = 1 - 4(2)(3)$

< 0

$\therefore$ no real roots (1)

(ii) (I) $\alpha + \beta = \frac{1}{2}$ (1)

(II) $\alpha\beta = \frac{3}{2}$ (1)

(III) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= \frac{1}{4} - 3$

$= -2\frac{3}{4}$ (1)

(IV) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$

$= \frac{1}{3}$ (1)

(iii) $\left(x - \frac{1}{\alpha}\right)\left(x - \frac{1}{\beta}\right) = 0$

$x^2 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)x + \frac{1}{\alpha\beta} = 0$ (1)

$\left.\begin{array}{l} \therefore x^2 - \frac{1}{3}x + \frac{2}{3} = 0 \\ \therefore 3x^2 - x + 2 = 0 \end{array}\right\}$ (1) either

(b) $\underbrace{40^c + 80^c + \ldots\ldots}_{16 \text{ months}}$ (1)

$= 0.40 \left(1 + 2 + \cdots + 2^{15}\right)$

$= 0.40 \left(\frac{2^{16} - 1}{1}\right) \approx \$26,214$ (1)

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(8) (a) $x = 40 + 10t - 5t^2$

$v = 10 - 10t$

$a = -10$

(i) At $t = 0$

$x = 40$ m (1)

(ii) At $x = 0$

$5t^2 - 10t - 40 = 0$

$\therefore t^2 - 2t - 8 = 0$

$(t-4)(t+2) = 0$

$\therefore t = 4$ secs (1)

(iii) At rest $v = 0$

$\therefore t = 1$ sec (1)

(iv) { acceleration is constant / UNIFORM } (1)

(v) (α) At $t = \frac{1}{2}$, $v = 5$ m/s (1)

At $t = 2$, $v = -10$ m/s (1)

(vi) { It turns around / it stops and turns around / changes direction } ANY (1)

(b) $4\sin^2\theta = 3$

$\sin\theta = \frac{\pm\sqrt{3}}{2}$ ← (1)

$\therefore \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

(1) (1) (1)

(c) $\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots$

$= \frac{2/3}{1 - 2/3}$

$= 2$ (1)

(9) (a) $SA = \pi r^2 + 2\pi r h = 75\pi$ ← (1)

(i) $\therefore h = \frac{75\pi - \pi r^2}{2\pi r}$

$= \frac{75 - r^2}{2r}$ ← (1)

(ii) $V = \pi r^2 h$

$= \pi r^2 \left(\frac{75 - r^2}{2r}\right)$

$= \frac{\pi r(75 - r^2)}{2}$ (1)

(iii) $\frac{dV}{dr} = \frac{75\pi}{2} - \frac{3\pi r^2}{2}$ (1)

$\frac{d^2V}{dr^2} = -\frac{6\pi r}{2}$

At MAX

$\frac{dV}{dr} = 0$ (1)

$\therefore 25 = r^2$

$\therefore$ { $r = 5$ } (1)

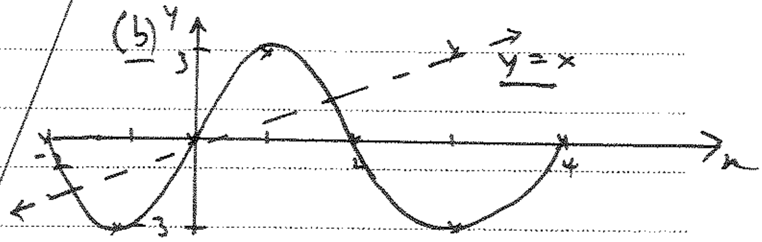
{ $h = 5$ }

{ $V'' < 0$ } ← (1)

$\Rightarrow$ MAXIMUM. $V = \frac{5\pi(50)}{2}$

(1) → $= 125\pi$ cc

(b)



(1) for $y = 3\sin\frac{\pi x}{2}$

(1) for drawing $y = x$

(1) for getting $y = x$ as the line.

(iii) 3 SOLUTIONS (1)

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Question 10:

(a)(i) $A = \frac{1}{3} \times \frac{1}{2} \times \left[1 + 4\left(\frac{2}{3}\right) + \frac{1}{2}\right]$ (1)

$= \frac{25}{36}$ (1)

(ii) $\int_1^2 \frac{1}{x}\,dx = \ln x\Big]_1^2$ (1)

$= \ln 2 - \ln 1$

$= \ln 2$ (1)

(iii) Areas approximate:

$\therefore \frac{25}{36} \approx \log 2$ (1)

$\therefore 2 = e^{25/36}$

$\therefore e = 2^{36/25}$

≈ 2.7132 (1)

(b) $y = x^2 \ln x$

$\frac{dy}{dx} = (\ln x)2x + x^2 \cdot \frac{1}{x}$

$= x[2\ln x + 1]$ (1) or equivalent

$\frac{d^2y}{dx^2} = (2\ln x + 1) + x\left(\frac{2}{x}\right)$

$= 2\ln x + 3$ (1)

For concave up $\frac{d^2y}{dx^2} > 0$

$\therefore 2\ln x + 3 > 0$

$\ln x > -\frac{3}{2}$

$x > e^{-3/2}$ (1)

(c) $\frac{d}{dx} e^{\cos x} = -\sin x\, e^{\cos x}$ (1)

$\therefore \int_0^{\pi/2} \sin x\, e^{\cos x}\,dx = \left[-e^{\cos x}\right]_0^{\pi/2}$ (1)

$= -e^{\cos \pi/2} + e^{\cos 0}$

$= -1 + e^1$

$= e - 1$ (1)